Elliptic Function Representation of Doubly Periodic Two-Dimensional Stokes Flows

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Abstract

We construct doubly periodic Stokes flows in two dimensions using elliptic functions. This method has advantages when the doubly periodic lattice of obstacles has less than maximal symmetry. We find the mean flow through an arbitrary lattice in response to a pressure gradient in an arbitrary direction, and show in a typical example that the shorter of the two period lattice vectors is an "easy direction" for the flow, an eigenvector of the conductance tensor corresponding to maximal conductance.

It is known, and we rederive it below, that two-dimensional (2D) Stokes flows can be represented in terms of two complex analytic functions [1]. It is plausible, then, that doubly periodic 2D Stokes flows should have a representation in terms of doubly periodic complex analytic functions, that is, elliptic functions [2, 3]. Such a representation was promised in 1959 by H. Hasimoto [4], but it did not appear. Other authors in the succeeding decades alluded to such a representation [5], and even, like Hasimoto, quoted results following from it [6]. The Hasimoto article may have appeared much later as lecture notes in Japanese [7]. Meanwhile, there are other, perhaps more straightforward, ways to represent doubly periodic 2D Stokes flows. These include matching of flows around a single obstacle across periodic cell boundaries [6, 8, 9, 10, 11], integral equation methods [12], biharmonic solvers on a grid [5], and finite element methods.

In spite of this long history, we have thought it useful to present the elliptic function method, because there is still, apparently, no readily available description of it. Furthermore, as we shall show, this approach solves one aspect of the problem which is not at all simple in the most common cell matching approach, namely the appropriate boundary condition for flow through a general periodic lattice in a general direction. With this method we describe the typical flow through a generic lattice.

1 2D Stokes Flows

Let us represent the 2D flow with velocity vector field $\vec{u}(x,y)$ as a complex scalar function u by means of the usual isomorphism

$$\vec{u} = u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}} \quad \leftrightarrow \quad u(x,y) = u_x + iu_y \tag{1}$$

Regarding the x-y plane as the complex z plane, we note that the divergence and curl of u are given by

$$\operatorname{div}\vec{u} \;=\; 2\Re(\partial u/\partial z) = 0 \tag{2}$$
$$\operatorname{curl}\vec{u} \;=\; 2\Im(\partial u/\partial z) = \omega \tag{3}$$

Here the first equation expresses incompressibility of the flow u, and the second defines the vorticity ω, understood as the (scalar) component of a vector field normal to the plane. Similarly, the gradient of the pressure P in this complex representation is

$$\vec{\nabla}P \quad \leftrightarrow \quad 2\partial P/\partial\overline{z} \tag{4}$$

The Stokes equation, representing the balance of viscous stress in the fluid by the pressure gradient, then becomes

$$\mu\nabla^2\vec{u} = \vec{\nabla}P \quad \leftrightarrow \quad 4\mu\,\partial^2 u/\partial\overline{z}\partial z \;= 2\partial P/\partial\overline{z} \tag{5}$$

where μ is the viscosity. In light of Eqs. (2) and (3) this means that

$$\partial(P/\mu - i\omega)/\partial\overline{z} = 0 \tag{6}$$

that is, that

$$f(z) = P/\mu - i\omega \tag{7}$$

is a single-valued function of z, holomorphic except for possible poles [1]. If we choose such a function f, we can integrate Eqs. (2) and (3) to find the flow

$$u = \frac{1}{4}(z\overline{f} - \int^z f\,dz + \overline{g}) \tag{8}$$

where $g(z)$ is a second holomorphic function, having logarithmic singularities at the poles of f. Thus u is represented in terms of two holomorphic functions, f and g.

The force exerted by the flow u on a finite obstacle, given by closed contour C, just involves the enclosed residues of f. The force on a small, oriented line segment Δz due to the fluid on its right is

$$\Delta F = iP\Delta z + 2i\mu(\partial u/\partial\overline{z})\Delta\overline{z} \tag{9}$$

Now, using Eqs. (7) and (8), and integrating over the closed curve C, oriented in the conventional positive direction, we find the force on C due to the fluid outside it

$$
\begin{aligned}
F &= \frac{i\mu}{2}\left[\oint_C (f+\overline{f})\,dz + \oint_C (z\overline{f'}+\overline{g'})\,d\overline{z}\right] && (10)\\
&= \frac{i\mu}{2}\left[\oint_C d(z\overline{f}+\overline{g}) + \oint_C f\,dz\right] && (11)\\
&= i\mu\oint_C f\,dz = -2\pi\mu\sum_C \mathrm{res}(f)\,. && (12)
\end{aligned}
$$

The last line follows because u in Eq. (8) is single-valued.

2 Pressure in Doubly Periodic Flows

Consider the integer lattice generated by two complex numbers, ω_1 and ω_3, with $\Im(\omega_3/\omega_1) > 0$, consisting of the points

$$W_{mn} = 2m\omega_1 + 2n\omega_3 \tag{13}$$

for all integers m and n. Suppose identical obstacles are located at these places, forming a doubly periodic array. A Stokes flow through this array, represented as in Eq. (8), is characterized by a function f with very restrictive properties. It is single-valued, it is holomorphic outside the obstacles, it has poles inside the obstacles, its imaginary part is doubly periodic, and its real part is a doubly periodic function plus a linear function, where the linear function is essentially the average pressure $< P >$, increasing linearly along its (constant) gradient. If we further ask for the simplest function of this type, having only simple poles at the W_{mn}, then there is essentially only one possibility, the Weierstrass zeta function $\zeta(z)$, with a linear correction. This follows from the theory of elliptic functions [2]. (Note that ζ depends also on the lattice constants ω_1 and ω_3, but we regard these as fixed parameters and do not indicate this dependence.)

More precisely we make use of the quasi-periodicity of $\zeta(z)$,

$$\zeta(z+2\omega_\alpha) = \zeta(z) + 2\eta_\alpha \tag{14}$$

for $\alpha = 1, 3$, where $\eta_\alpha = \zeta(\omega_\alpha)$ are constants satisfying

$$\eta_1\omega_3 - \eta_3\omega_1 = i\pi/2 \tag{15}$$

Then we can take, for the function f, either of

$$f_\alpha = \frac{-i}{|\omega_\alpha|}\left[\omega_\alpha\zeta(z) - \eta_\alpha z\right], \tag{16}$$

where $\alpha = 1, 3$. Using Eqs. (14) and (15), we verify that

$$
\begin{aligned}
f_1(z+2\omega_1) &= f_1(z) && (17)\\
f_1(z+2\omega_3) &= f_1(z) - \pi/|\omega_1| && (18)\\
f_3(z+2\omega_1) &= f_3(z) + \pi/|\omega_3| && (19)\\
f_3(z+2\omega_3) &= f_3(z) && (20)
\end{aligned}
$$

Thus in both functions only the real part, which we interpret as P/μ, shows a linear growth. The average pressure gradient corresponding to f_α is perpendicular to ω_α. From its projection on the other lattice vector we determine that it is

$$< \vec{\nabla} P > \quad \leftrightarrow \quad 2 < \partial P/\partial \overline{z} >= \frac{-i\omega_\alpha \mu \pi}{2|\omega_\alpha| \Im(\overline{\omega_1}\omega_3)} \tag{21}$$

By taking linear combinations of f_1 and f_3 we can find a suitable f with the pressure gradient in any direction with respect to the lattice. The combination corresponding to a pressure gradient of the same magnitude as f_1 and f_3 but in the direction $e^{i\theta}$ is

$$f = e^{i\theta}\zeta - Cz \tag{22}$$

with

$$C = i\left(\frac{\eta_1 \Re(e^{-i\theta}\omega_3) - \eta_3 \Re(e^{-i\theta}\omega_1)}{\Im(\overline{\omega}_1 \omega_3)}\right) \tag{23}$$

The force on the fluid in a period parallelogram D, with sides $2\omega_1$ and $2\omega_3$, due to the average pressure in the flow corresponding to f_α, is

$$\begin{aligned} F_P &= \oint_{\partial D} < P > i dz = i \int\int_D < \partial P/\partial \overline{z} > d\overline{z} dz & (24) \\ &= -2 < \partial P/\partial \overline{z} > \int\int_D dx\, dy = \frac{2i\omega_\alpha \mu \pi}{|\omega_\alpha|} & (25) \end{aligned}$$

The force on the fluid due to each obstacle in this same flow, since $\zeta(z)$ is normalized to have residue 1 at each simple pole, is $-2i\omega_\alpha \mu\pi/|\omega_\alpha|$, by Eq. (12). There is on average one obstacle in each period parallelogram. Thus the force on the fluid due to the average pressure (applied somehow from outside) is balanced by the force due to the obstacles, and the net force on the fluid is zero, as is always the case in Stokes flows.

A more general function f in the representation of Eq. (8) for the Stokes flow u would be a superposition of translates of f_1 and f_3, always keeping the singularities inside the obstacles, and not in the physical region of flow. Equivalently, one could take a multipole expansion of such functions. This would be a series in derivatives of f_1 and f_3. These functions are doubly periodic with higher order poles at the lattice points, that is, they are the Weierstrass $\mathcal{P}$ function and its derivatives. Thus f has the form

$$f(z) = e^{i\theta}\zeta - Cz + \sum_n^{N_c} c_n \mathcal{P}^{(n)} \tag{26}$$

If the boundary contours have reflection symmetry in the origin, then it is enough to take only the terms in the series with n odd.

3 Velocity Field in Doubly Periodic Flows

From Eq. (8) we know that the flow u corresponding to f above is

$$u = \frac{1}{4}\left(ze^{-i\theta}\overline{\zeta} - \overline{C}|z|^2 - 2e^{i\theta}\ln|\sigma| - Cz^2/2 + z\sum_n^{N_c}\overline{c}_n\overline{\mathcal{P}^{(n)}} - \sum_n^{N_c} c_n\mathcal{P}^{(n-1)} + \overline{g}\right) \tag{27}$$

where g is a second analytic function, still to be determined. We have already used the freedom to add terms of the form $\overline{g}$ in adding a term $e^{i\theta}\ln\overline{\sigma}$, where σ is the sigma function, another of the special functions of elliptic function theory [2]. Such logarithmic terms compensate the multivaluedness of logarithmic terms in the integral of f.

Now among all the functions g that we could choose, we want the one that makes u doubly periodic, and that makes $u = 0$ (say) on the boundary contours of the obstacles. Since the logarithmic part of g has been explicitly written as a separate term, g as defined above is single-valued, and hence has a Laurent series in a neighborhood of the origin. We must anticipate, though, that g has singularities at all the lattice points W_{mn}, and hence that this series would converge only out to the nearest such lattice point. This difficulty can be removed because double periodicity of u determines the principal part of g at all lattice points. We see, for example, that u blows up logarithmically at $z = 0$ but, because of the poles in ζ, it blows up like $1/z$ at every other lattice point. This is inconsistent with double periodicity, and hence g must cancel the $1/z$ behavior at all W_{mn} except 0. Also, in the Laurent series for g about $z = 0$, there will be negative powers of z. Translates of these terms contribute to the principal part of g at other W_{mn}. Let S be a set of indices (m, n) sufficiently large to label all W_{mn} in a disk large enough to contain a period parallelogram. Then we take g in the form

$$g = -\sum_{(m,n)\in S}\frac{\overline{W}_{mn}}{z - W_{mn}} + \sum_{j=0}^{N_b} b_j \sum_{(m,n)\in S}(z - W_{mn})^{-j} + \sum_{j=1}^{N_a} a_j z^j \tag{28}$$

Because the poles of g have been included as explicit terms, the power series with coefficients a_j now converges out to the nearest W_{mn} not indexed in S, and we choose this index set large enough to get good numerical behavior in our final step.

Lastly we specify the boundary conditions that determine the coefficients a_n, b_n, and c_n in Eqs. (27) and (28). We choose a region D with a boundary ∂D that is entirely within the fluid region, and such that every point z in ∂D has a corresponding point z' in ∂D displaced by one of the lattice constants $\pm 2\omega_\alpha$. Typically D would be the period parallelogram centered on the origin with vertices $(\pm\omega_1, \pm\omega_3)$. We ask that the complex function u be periodic, taking the same value at points z and z' in ∂D related by a lattice constant, and also that the derivative $\partial u/\partial\overline{z}$ be periodic in the same sense. We also require that u take prescribed values on the boundary contour C of the obstacle. Choosing a large number of pairs of points (z, z') on ∂D and also points z on C for imposing these conditions, we obtain an overdetermined inhomogeneous system of linear equations that we then solve in the sense of least squares. Because c_n occurs together with its complex conjugate $\overline{c_n}$, the system must be regarded as real linear, not complex linear. With suitable choices for the truncation parameters N_a, N_b, and N_c, this boundary value problem has a solution accurate to many decimal places. We confirm the solutions in [6], for flows through square and triangular lattices of obstacles, to 5 decimal places, typically.

It is superficially surprising that one can stipulate conditions on both u and its derivative in this elliptic problem, but it is clear physically that such periodic solutions must exist. It is also clear mathematically, if one considers the equivalent Stokes flow on a torus. This argument only applies, though, if the applied stress that drives the flow has all the properties that we have required of the function f in Section 2. For other functions f the boundary value problem would have no solution. Knowing the form of f in advance, as we do, the actual boundary values of P and u emerge as part of the solution. Other approaches to this problem are typically restricted to situations in which one knows the boundary values by symmetry, where P is constant on part of ∂D, for example, or where u is normal or tangential on ∂D, special cases that do not hold for general lattices and flows.

Let us define the *mean* of a flow u to be the constant flow V such that the flux of V and the flux of u into a period parallelogram are the same. By incompressibility of the flow, this is also the flux of V (and u) out of a period parallelogram. If we take the period parallelogram to be the one with vertices $(\pm\omega_1, \pm\omega_3)$, then we have the formula

$$V = \frac{1}{2\Im(\overline{\omega_1}\omega_3)} \left[\Im\left(\int_{\omega_1-\omega_3}^{\omega_1+\omega_3} \overline{u}\, dz \right) \omega_1 + \Im\left(\int_{\omega_1+\omega_3}^{-\omega_1+\omega_3} \overline{u}\, dz \right) \omega_3 \right] \tag{29}$$

The integrals depend only on the endpoints, by incompressibility of the flow, and the formula is invariant under a common translation of the endpoints. Thus the choice of period parallelogram is in fact arbitrary.

Using this formulation we have computed the tensorial relation between the force F in Eq. (12) and the mean velocity V in Eq. (29) in many examples. For regular square and triangular lattices, F and V are simply proportional, but more generally

$$V = GF \tag{30}$$

where the conductance tensor G is a real symmetric matrix (and we are now representing V and F as real 2-vectors, not complex scalars). A typical computed flow is shown in Figure 1. Here $\omega_1 = 1$, $\omega_3 = 0.2 + 0.5i$, and the circular obstacles have radius 0.2. The pressure gradient F is in the y direction, but the mean flow V is approximately along ω_3, the shorter of the lattice vectors. The reason for this behavior is simple, and in accord with common sense. The direction of ω_3 is an "easy direction" for the flow, an eigenvector of G, and the corresponding conductance eigenvalue for the rather wide "channels" along this direction is large. The obstacles, being relatively closely spaced along the direction of ω_3, create "walls" along the sides of the channels. The conductance

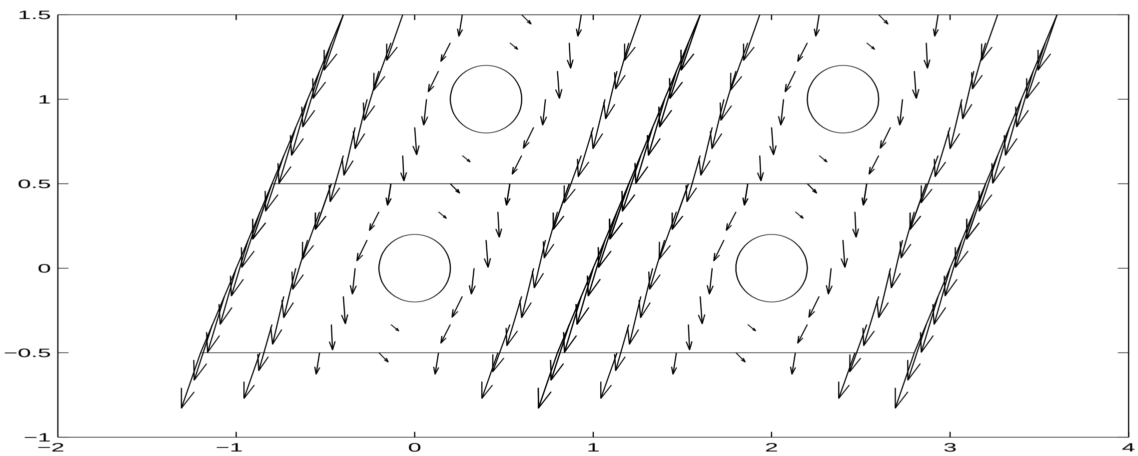

Figure 1: A Stokes flow through a lattice of circular obstacles is shown in four period parallelograms. The lattice vectors are $\omega_1 = 1$, $\omega_3 = 0.2 + 0.5i$, and the obstacles have radius 0.2. The pressure gradient is vertical, but the mean flow is, to good approximation, along ω_3, the shorter of the lattice vectors.

corresponding to the other, perpendicular eigenvector, normal to the "walls", is much smaller than the conductance along the channels. The ratio of the two eigenvalues, a measure of the anisotropy of the conductance tensor, is about 2.545 in this example, noticeably more anisotropic than the lattice itself. It is clear that the conductance anisotropy could be indefinitely large for this lattice if the round obstacles were larger and almost touched along the ω_3 direction.

The theory described here provides a starting point for problems involving flow near ciliated surfaces. Within a ciliated layer, the cilia, hairlike projections from the surface, would constitute the regular array of obstacles. In the simplest case they would be rigid and perpendicular to the surface. Consider, for example, a straight pipe of circular cross section, with such a ciliated inner surface, still leaving a free cylindrical channel down the center. Suppose the length of the cilia is much greater than the mean distance between them, and much less than the radius of the pipe. The Poiseuille flow problem asks for the flow in response to a pressure difference between one end of the pipe and the other. The usual boundary condition says that the flow should be zero on the walls of the pipe, and this condition still determines the flow in a thin region near the wall, comparable in thickness to the mean spacing of the cilia, but within most of the ciliated layer, the mean flow will be given in the manner described here. Via a similarly thin transition layer, the Poiseuille flow down the unobstructed central channel matches this mean flow at its edge. Systematic anisotropy in the placement of the cilia, including, one must anticipate, a merely statistical anisotropy, would impart rotation to the fluid channel, as it amounts to a “rifling” of the inner surface of the pipe. Considerations of this kind are relevant in biological settings where such surfaces are very common. The cilia as we have just described them are merely passive obstacles, but often they are active agents driving flows. The methods of this paper could also be a starting point for treating such phenomena, as we hope to show in the future.

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References

[1] J. Happel and H. Brenner, **Low Reynolds Number Hydrodynamics**, Prentice-Hall, Englewood Cliffs, NJ, 1965, p. 59.

[2] Lawden, D.F., **Elliptic Functions and Applications**, Springer-Verlag, New York, NY, 1989.

[3] M. Abramowitz and I.A. Stegun, **Handbook of Mathematical Functions**, Dover Publications, New York, NY, 1965.

[4] H. Hasimoto, "On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres," **J. Fluid Mech. 5** (1959), 317-328.

[5] M.A. Zaks and A.V. Straube, "Steady Stokes Flow with Long-Range Correlations, Fractal Fourier Spectrum, and Anomalous Transport," **Phys. Rev. Lett. 89** (2002), 244101.

[6] A.S. Sangani and A. Acrivos, "Slow Flows Past Periodic Arrays of Cylinders with Application to Heat Transfer," **Int. J. of Multiphase Flow 8** (1982), 193-206.

[7] H. Hasimoto, **Lect. Notes Res. Inst. Math. Sci. Kyoto Univ.**, No. 214 (1974).

[8] M.V. Bruschke and S.G. Advani, "Flow of Generalized Newtonian Fluids Across a Periodic Array of Cylinders," **Journal of Rheology 37** (1993), 479-498.

[9] C.Y. Wang, "Stokes Flow Through an Array of Rectangular Fibers," **Intl. J. of Multiphase Flow 22** (1996), 185-194.

[10] M. Hellou, J. Martinez, and M. El Yazidi, "Stokes Flow through Microstructural Model of Fibrous Media," **Mechanics Research Communications 31** (2004), 97-103.

[11] V.A. Kirsh, "Stokes Flow in Periodic Systems of Parallel Cylinders with Porous Permeable Shells," **Colloid Journal 68** (2006), 173-181.

[12] L. Greengard and M.C. Kropinski, "Integral Equation Methods for Stokes Flow in Doubly Periodic Domains," **Journal of Engineering Mathematics 48** (2004), 157-170.